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The Characteristic Equation of the Euler-Cauchy Differential Equation and its Related Solution Using MATLAB

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Abstract **- The behavior of nature is usually modelled with Differential Equations in various forms. Depending on the constrains and the accuracy of a model, the connected equations may be more or less complicated. For simple models we may use Non-Homogeneous Equations but in general, we have to deal with Homogeneous ones since from a physicists point of view nature seems to be Homogeneous. In many applications of sciences, for solving many of them, often appear equations of type n^{th} order Linear differential equations, where the number of them is Euler-Cauchy differential equations. i.e. Euler-Cauchy differential equations often appear in analysis of computer algorithms, notably in analysis of quick sort and search trees; a number of physics and engineering applications. In this paper, the researcher aims to present the solutions of a homogeneous Euler-Cauchy differential equation from the roots of the characteristics equation related with this differential equation using MATLAB. It is hoped that this work can contribute to minimize the lag in teaching and learning of this important Ordinary Differential Equation.**

Keywords: **Euler-Cauchy Differential Equation, Ordinary Differential Equation, Characteristics Equation, MATLAB**

I. INTRODUCTION

Differential Equations serve as Mathematical models for many exciting real-world problems in Science and Technology and also in different fields such as Economics, Medicine, Ecology, etc. [1] Many of the principles, or laws, that rule the behavior of the physical world are propositions, or relationships involving the rate at which things happen. Thus the theory of Differential Equations has a prominent place in the Mathematics curricula all over the world and is extensively used in research in almost every branch of knowledge.

Equations containing derivatives are called Differential Equations. The Ordinary Differential Equations $F\left(x, y, y', y'', \dots\dots\dots, y^{(n)}\right) = 0$ circumscribe a very broad area of Mathematics and of fundamental importance to explain various models of real life [1][2][3]. Geometrically, the general solution of a differential equation represents a family of curves that are called the integral curves. This solution is called integral or primitive of the differential equation. Solving a differential equation means finding an adequate family of curves. Many of the principle, or laws, from physics to biology, through the fields of medicine and engineering, can be described by means of these differential equations. The solutions of these equations are used, to identify population growth, construct buildings and bridges, explain electric circuits, to design automobiles, among many other applications. A single differential equation can serve as a Mathematical model for several different phenomena [4][6][7][8][9][12].

The purpose of this paper is to present the solution of an Ordinary Differential Equation, called the Euler-Cauchy Equation from the roots of the Characteristic Equation related with this differential equation using MATLAB. We first define the Homogeneous and Non-Homogenous Euler-Cauchy equation of order 'n' [1][2][3][10][11]. Then we will use the particular cases n=2, n=3 to present its solutions, according to the roots of its characteristic equation. In the following section, several equations, characteristics related with their Euler-Cauchy Equations will be presented, and finally a conclusion will be reported.

II. THE HOMOGENEOUS EQUATION OF EULER-CAUCHY

Leonhard Euler (1707-1783) was a Swiss mathematician, astronomer, physicist, geographer who worked in the area of Calculus and Graph Theory and spent much of his life in St. Petersburg and Berlin [5]. He also introduced much of the modern mathematical terminology and notations, particularly for mathematical analysis, such as the notion of a mathematical function. He is also known for his work in mechanics, fluid dynamics, optics, astronomy and music theory.

Augustin Louis Cauchy (1789-1857), French mathematician, pioneered the study of analysis, both real and complex, and the theory of permutation groups [5]. He also researched in convergence and divergence of infinite series, differential equations, determinants, probability and mathematical physics.

The homogeneous equation of Euler-Cauchy of n^{th} order is any Ordinary Differential Equation of the form

$$\alpha_0 x^n y^{(n)} + \alpha_1 x^{n-1} y^{(n-1)} + \cdots + \alpha_{n-1} x y^{()} \alpha_n y = 0 \quad (\alpha_0 \neq 0) \qquad (1)$$

where $\alpha_0, \alpha_1, \alpha_2, \dots, \alpha_n$ are constants.

In this case the solution will be some function whose first derivative product by x, the second derivative product by x^2, the third derivative multiplied by x^3, and so on, up to the n^{th} derivative product by x^n being linearly dependent on the original function.

The function that has this property is the function:

$$y = x^m \quad (2)$$

in which 'm' it is a real constant whatever.

III. STUDY OF THE SOLUTIONS OF THE EULER-CAUCHY EQUATION

In this section, we consider the second order Euler-Cauchy equation $\alpha_0 x^2 y^{('')} + \alpha_1 x y^{(')} + \alpha_2 y = 0$. Always, mathematicians have crafted in genious methods to solve some very specialized equations, therefore there are not surprisingly, a large number of differential equations that can be solved analytically [4].

To solve the Euler-Cauchy equation of type:

$$\alpha_0 x^2 y^{('')} + \alpha_1 x y^{(')} + \alpha_2 y = 0 \quad (3)$$

we must look for a solution of the form
$y = x^m$ (2). Substituting (2) into (3), we get:

$$\alpha_0 x^2 m(m-1)x^{m-2} + \alpha_1 x m x^{m-1} + \alpha_2 x^m = 0$$

Hence, $\alpha_0 m(m-1) + \alpha_1 m + \alpha_2 = 0$.

$\therefore \alpha_0 m^2 + (\alpha_1 - \alpha_0)m + \alpha_2 = 0$ is called the characteristic equation related with the Euler-Cauchy equation of the type $\alpha_0 x^2 y^{('')} + \alpha_1 x y^{(')} + \alpha_2 y = 0$. The solution of this Ordinary Differential Equation depends on the roots of its characteristic equation. The solution of the higher-order equation follows similarly.

We have three cases to consider for the solution of the characteristic equation

A. Case I: Roots are Real and Distinct

If the roots are m_1 and m_2 of the characteristic equation, then the solution of the Euler-Cauchy equation is of the form:

$y = Ax^{m_1} + Bx^{m_2}$, where A and B are arbitrary constants.

B. Case II: Roots are Real and Equal

If the roots are equal, supposing $m_1 = m_2 = m$ are the roots of the characteristic equation, then the solution of the Euler-Cauchy equation is of the form:

$y = (A + B\, In\, x)x^m$, where A and B are arbitrary constants.

C. Case III: Roots are Complex Conjugate

If the roots m_1, m_2 of the characteristic equation are complex conjugate, i.e. $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$ where α and β are real with $\beta \neq 0$, then the solution of the Euler-Cauchy equation is of the form:

$$y = x^{\alpha}[A\cos(\beta\, In\, x) + B\sin(\beta\, In\, x)],$$

where A and B are arbitrary constants.

IV. SOME EQUATIONS CHARACTERISTICS ASSOCIATED WITH THE EULER-CAUCHY EQUATION AND EXAMPLES

In this section, for each Homogeneous Equation of Euler-Cauchy of n^{th} order (Table I), we will present, respectively, its characteristic equation that will be a polynomial equation of degree 'n'. Then Five examples will be discussed, one for each case.

TABLE I THE EULER-CAUCHY EQUATION HOMOGENEOUS ASSOCIATED WITH ITS CHARACTERISTIC EQUATION (UNTIL FIFTH ORDER)

Euler-Cauchy Differential Equation	Characteristic Equation
$\alpha_0 x y^{(')} + \alpha_1 y = 0$	$\alpha_0 m + \alpha_1 = 0$
$\alpha_0 x^2 y^{('')} + \alpha_1 x y^{(')} + \alpha_2 y = 0$	$\alpha_0 m^2 + (\alpha_1 - \alpha_0)m + \alpha_2 = 0$
$\alpha_0 x^3 y^{(''')} + \alpha_1 x^2 y^{('')} + \alpha_2 x y^{(')} + \alpha_3 y = 0$	$\alpha_0 m^3 + (\alpha_1 - 3\alpha_0)m^2 + (2\alpha_0 - \alpha_1 + \alpha_2)m + \alpha_3 = 0$
$\alpha_0 x^4 y^{(iv)} + \alpha_1 x^3 y^{(''')} + \alpha_2 x^2 y^{('')} + \alpha_3 x y^{(')} + \alpha_4 y = 0$	$\alpha_0 m^4 + (\alpha_1 - 6\alpha_0)m^3 + (11\alpha_0 - 3\alpha_1 + \alpha_2)m^2 + (2\alpha_1 - 6\alpha_0 - \alpha_2 + \alpha_3)m + \alpha_4 = 0$
$\alpha_0 x^5 y^{(v)} + \alpha_1 x^4 y^{(iv)} + \alpha_2 x^3 y^{(''')} + \alpha_3 x^2 y^{('')} + \alpha_4 x y^{(')} + \alpha_5 y = 0$	$\alpha_0 m^5 + (\alpha_1 - 10\alpha_0)m^4 + (35\alpha_0 - 6\alpha_1 + \alpha_2)m^3 + (11\alpha_1 - 50\alpha_0 - 3\alpha_2 + \alpha_3)m^2 + (24\alpha_0 - 6\alpha_1 + 2\alpha_2 - \alpha_3 + \alpha_4)m + \alpha_5 = 0$

Example-1: Given the Euler-Cauchy equation:
$3xy^{(')} - 2y = 0$

Characteristic equation: $3m - 2 = 0$
MATLAB: $m = \frac{2}{3}$
Solution: $y = Ax^{\frac{2}{3}}$

Example-2: Given the Euler-Cauchy equation:
$x^2y^{('')} + 7xy^{(')} + 9y = 0$

Characteristic equation: $m^2 + 6m + 9 = 0$
MATLAB: $m_1 = -3$ and $m_2 = -3$
Solution: $y = (A + B\, In\, x)x^{-3}$

Example-3: Given the Euler-Cauchy equation:
$x^3y^{(''')} - 3x^2y^{('')} + 6xy^{(')} - 6y = 0$

Characteristic equation:
$$m^3 - 6m^2 + 11m - 6 = 0$$
MATLAB: $m_1 = 1, m_2 = 2, m_3 = 3$

Solution: $y = Ax + Bx^2 + Cx^3$

Example-4: Given the Euler-Cauchy equation:
$$x^4y^{('v)} + x^3y^{(''')} - x^2y^{('')} - 2xy^{(')} + 6y = 0$$

Characteristic equation:
$$m^4 - 5m^3 + 7m^2 - 5m + 6 = 0$$

MATLAB:
$m_1 = 2, m_2 = 3, m_3 = +i, m_4 = -i$

Solution:
$y = Ax^2 + Bx^3 + C\,\mathrm{Cos}(\mathrm{In}\,x) + D\,\mathrm{Sin}(\mathrm{In}\,x)$

Example-5: Given the Euler-Cauchy equation:
$$4x^5y^{(v)} + 24x^4y^{('v)} + 53x^3y^{(''')} + 20x^2y^{('')} + 45xy^{(')} - 45y = 0$$

Characteristic equation:
$$4m^5 - 16m^4 + 49m^3 - 75m^2 + 83m - 45 = 0$$

MATLAB: $m_1 = 1, m_2 = \frac{1}{2} + i\sqrt{2},$
$$m_3 = \frac{1}{2} - i\sqrt{2}, m_4 = 1 + i2, m_5 = 1 - i2$$
Solution: $y = Ax + x^{\frac{1}{2}}\left[B\,\mathrm{Cos}\left(\mathrm{In}\left(x^{\sqrt{2}}\right)\right) + C\,\mathrm{Sin}\left(\mathrm{In}\left(x^{\sqrt{2}}\right)\right)\right] + x\left[D\,\mathrm{Cos}\left(\mathrm{In}\left(x^2\right)\right) + E\,\mathrm{Sin}\left(\mathrm{In}\left(x^2\right)\right)\right]$

V. ALGORITHMS IN MATLAB

If the coefficients $\alpha_0, \alpha_1, \alpha_2, \ldots, \alpha_n$ in a homogeneous Euler-Cauchy differential equation of order n:
$\alpha_0 x^n y^{(n)} + \alpha_1 x^{n-1} y^{(n-1)} + \cdots + \alpha_{n-1} x y^{(')} \alpha_n y = 0$
verify in the conditions

$$\alpha_{k-n} = \frac{(k-n+1)}{k.n}\,\frac{\alpha_{k-n+1}}{\alpha_n}\,(\alpha_{n-1} - n(k-1)\alpha_n)$$
where k = 2,3,4,....,n. (4)

Then, the solution of the equation is
$$y = x^{-\frac{\alpha_{n-1}}{n\alpha_n}}(A_1x^{n-1} + \cdots + A_1x + A_0) \qquad (5)$$

A. Algorithm for Finding the Coefficients

Algorithm for finding the coefficients in equation (4) are true, and as follows

First, in MATLAB ⇒Menu "File" ⇒ Select "NEW" ⇒ Select "M-File" ⇒ The following function to copy in file, and then save the file to "euler_fun.m" name

```
function euler_fun(a)
syms n k
n=a{3};
for k=2;n
a{k+1}=(((n-k+1)*a{k}*(a{2}-n*(k-1)*a{1}))/(n*k*a{1}));
disp(sprintf('a(%d)',n-k));
display (simplify(a{k+1}));
end
end
```

Secondly, in the window "Command Windows" of MATLAB, type the following commands:

```
>>f{1}="Coefficient: α0";
>>f{2}="Coefficient: α1";
>>f{3}="Order of differential equation : n";
>>euler_fun(f)
```

Example-6: In the window "Command Windows" of MATLAB, type the following commands:

```
>>f{1}=1;
>>f{2}=-3;
>>f{3}=3;
>>euler_fun(f)
```

Then, key "ENTER"

```
a(1)
ans = 6
a(0)
ans = -6
```

Thus, functions that are true in the equation (4), are calculated, the desired equation is as follows:

$x^3y^{(''')} - 3x^2y^{('')} + 6xy^{(')} - 6y = 0$
i.e. The solution is $y = Ax + Bx^2 + Cx^3$

B. Algorithm to Check the True Coefficients

Algorithm to check the true coefficients in equation (4), is as follows

First, in MATLAB ⇒ Menu "File" ⇒ Select "New" ⇒ Select "M-File" ⇒ The following function to copy in file, and then save the file to "euler_true.m" name:

```
function euler_true(a)

syms n k
n=size(a,2)-1;
k=2;
flag=1;
```

```
while(flag==1 && k<=n)
if(abs((a{n-k+1}*(n*k*a{n+1}))-((n-k+1)*a{n-
k+2}*(a{n}-n*(k-1)*a{n+1})))<=1e-012)
flag=1;
k=k+1;
else
flag=0;
end
if(flag==1)
display('Yes');
else
display('No');
end
end
```

Secondly, in the window "Command Windows" of MATLAB, type the following commands:

>>f{1}="The calculated Coefficient : α_0";
>>f{2}="The calculated Coefficient: α_1";
>>.........................
>>f{n}="The calculated Coefficient: α_{n-1}";
>>f{n+1}="The calculated Coefficient: α_n";
>>euler_true(f)

Example-7: In the window "Command Windows" of MATLAB, type the following commands:

```
>>f{1}=6;
>>f{2}=18;
>>f{3}=9;
>>f{4}=1 ;
>>euler_true(f)
```

Then, key "ENTER"

>>yes

Thus, if $\alpha_0 = 1, \alpha_1 = 9, \alpha_2 = 18, \alpha_3 = 6$, equation (4) are established, i.e. equation

$$x^3 y^{(''')} + 9x^2 y^{('')} + 18xy^{(')} + 6y = 0$$

And the solution is

$$y = Ax^{-1} + Bx^{-2} + Cx^{-3}$$

VI. CONCLUSION

This paper is grown from a proposal to present the solution of a homogeneous Euler-Cauchy Equation from its related characteristic equation. The growth of the solution set of certain Ordinary differential Equations still remains the object of research, with enticing problems and high applicability in the phenomena of nature. Euler-Cauchy Differential Equations often appear in analysis of computer algorithms, notably in analysis of quick sort and search trees; a number of physics and engineering applications, such as when solving Laplace's equation in polar co-ordinates, and many other sciences. In the illumination of the above, it is expected that this work will significantly stimulate other research on Euler-Cauchy Equation in order to minimize the lag between Mathematical abstraction and its practice.

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