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The Technique Homotopy Perturbation Method Operated on Laplace Equation

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***Abstract* - In this study, we introduce a technique acknowledged as the Homotopy Perturbation Method (HPM) for obtaining the particular solution of two-dimensional Laplace's Equation with conditions like Dirichlet, Neumann and the use of different boundary prerequisites to exhibit this method's potential and reliability. The steady-state condition, which depends on temperature, converts Laplace's equation into a greater dimension and deforms the equal into a Partial Differential Equation (PDE). Here we additionally tried to discover a comparative measurement in terms of literature survey [1] between the results bought by means of the HPM approach and the same result for the identical equation introduced in any other technique eventually referred to as the Variable Separation Method (VSM). The consequences exhibit that HPM has excessive efficiency and effectiveness in fixing Laplace's equation. Also dealing without delay with the trouble has a wide variety of benefits and furnished the approximate solution which converges very unexpectedly to a correct answer.**
***Keywords:* Homotopy Perturbation Method (HPM), Partial Differential Equation, Laplace's Equations**

I. INTRODUCTION

In this literature we tried to enforce a choice as properly as a dependable manner for the answer of Partial Differential Equation (PDE) in linear and non-linear systems. Non-linear troubles can be solved by the use of perturbation strategies which suffer some restrictions due to requirement of a small parameter. If the preference of the small parameter is now not appropriate then it would be not easy to resolve the non-linear equations. Due to these boundaries, the perturbation approach wishes modifications. In unique, we proposed Homotopy Perturbation Method (HPM) for the answer of linear and non-linear PDE, which has been precisely examined on Helmholtz equation, Fisher's equation, Initial boundary fee hassle [2] and many more. The perturbation approach is blended with Homotopy which has a significant phase in differential topology. Hence, we get a new approach which is referred to as Homotopy Perturbation Method (HPM). This new technique was once advocated and introduced via J.H. He [3,4]. The approach used in HPM acts as a very essential perturbation rule [5] for fourth order PDE, structures of PDE, and higher-dimensional preliminary boundary fee problems. Iterative schemes keep away from discretization, linearization, and restrictive assumptions, as properly as round-off errors. With the assistance of this approach, a non-linear figure is deformed always into easily commutable traits and makes it simpler to deal with the same. Also, non-requirement of small parameters in an equation plays a vital gain that permits the technique to furnish approximate analytic formulation [6] broadly relevant to linear and non-linear issues in the utilized field of science and mathematics. HPM depends on a convergent collection without parameters and computable factors (convergent series) to supply a solution. Later the technique is additionally developed for finding the non-linear differential equation [7]. It has been proposed that the preference of the homotopy equation and approximation in initial stage must be appropriate [8] to produce the correct and convergent solution. The process can be carried out without problems and efficient to convert the approach into an effective technique. It is solely essential to set preliminary stipulations for the HPM, as antagonistic to organizing boundary prerequisites for the approach of separation of variables. The truth is that the proposed HPM solves non-linear issues barring the usage of Adomian's polynomials [9,10] which can be viewed as a clear benefit of the approach over the decomposition.

Apart from the mathematical factor of view, the applicative section for HPM in the fields such as non-linear fuel dynamic equation [11], Volterra's Integro-differential equation [12], bifurcation of non-linear troubles [13], non-linear oscillators [14], bifurcation of delay-differential equations [15], non-linear wave equations [16], boundary cost issues [17,18] are additionally intervening. Another benefit of the approach is that the answer converges hastily and we get the required the same after little reiteration [19]. The primary attribute of the current work is the implementation of the Homotopy Perturbation technique for fixing Laplace's equation. Laplace's equations nothing but a unique type of second-order partial differential equation and its set of selections R identified as harmonic aspects which enfolds the potentials of electrical, magnetic, and gravitational steady-state temperatures, as well as hydrodynamics. The equation considered as Laplace equation, used to be discovered with the useful resource of the French mathematician and astronomer Pierre-Simon Laplace (1749-1827). Laplace's equation states that the sum of the second-order partial derivatives of *R*, the unknown harmonic function, with the consciousness of the cartesian coordinates, equals zero. In particular, we additionally make a remedy for the hassle with the technique for separation of variables. A graphical approach (using MATLAB) [20] helps us to visualize an easy contrast amongst the approximate answer bought by using HPM, different perturbation methods, and additionally the genuine answer making use of the separation of variables approach (SVM). The gain of this techniques to deal immediately with the trouble and get the approximate answer which converges very swiftly to the correct values as solution with excessive efficiency and effectiveness. In the strive to estimate absolute error, the approach proves itself very fundamental in the history of error-free mathematical scenarios.

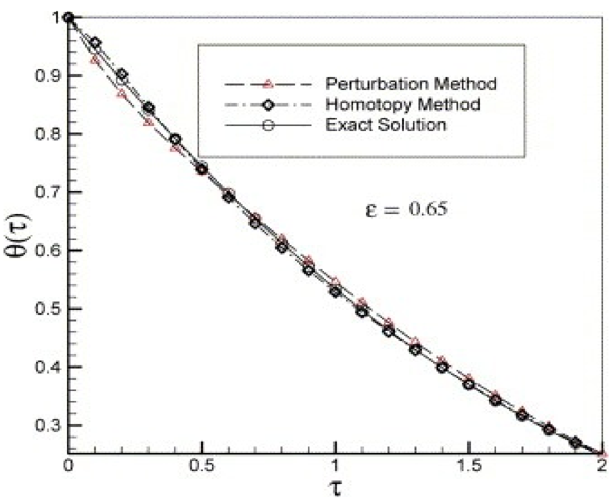

Fig. 1 Comparison Among Different Iterative Methods in Converging Towards Exact Solution [1]

II. DESCRIPTION OF HOMOTOPY METHOD AND ITS SERIES SOLUTION

What is Homotopy? Let, X, Y be two topological areas and f, g be two continuous functions from X to Y i.e., $f : X \rightarrow Y$, $g : X \rightarrow Y$. A Homotopy from f to g is another continuous function $F : X \times [0, 1] \rightarrow Y$ satisfying $F(x, 0) = f(x)$ and $F(x, 1) = g(x)$, for all $x \in X$. The technique was once first delivered via Professor Ji-Huan He [2,3] and following him, it is known as a homotopy with an embedding parameter $p \in [0, 1]$.

This approach makes use of a small parameter known as the embedding parameter for its construction, naming it the homotopy perturbation method, which has a range of blessings over current perturbation methods. To illustrate its effectiveness and its convenience, a Duffing equation [21] with excessive order of nonlinearity is used; the result displays that in spite of very massive parameter values, its first order of approximation generated by means of the proposed technique is legitimate uniformly and is extra correct with appreciate to the perturbation solutions.

Now we shall deliver the simple formalism of the homotopy perturbation method. The approach does now not comprise any calculation of a domain polynomial, any linearization or discretization, or any perturbation or transformation [22]. Here preliminary prerequisites are required to get an answer by the homotopy perturbation method. To supply quick thought of the method, let us think about a non-linear differential equation

$$A(u) - f(r) = 0, \; r \in \Omega \; (1) \quad \ldots\ldots\; (1)$$

with the boundary condition: $B(u, \frac{\delta u}{\delta n}) = 0, \; r \in \Gamma$

Here A = general differential operator, B =boundary operator,

f(r) = analytical function G = boundary of the domain Ω.

According to the structural formulation of the Homotopy technique,

A = L + N, L is linear and N is nonlinear.

The equation (1) takes the form $L(u) + N(u) - f(r) = 0, \; r \in \Omega$

We get the Homotopy function as follows

$$H(v,p) = (1-p)[L(v)-L(u_0)] + p[A(v) - f(r)] = 0, \quad \ldots\ldots(2)$$

where $v : \Omega[0, 1] \rightarrow R$

$p \in [0, 1]$ where [0, 1] is the embedding parameter, u_0 is the first approximation and satisfies the boundary conditions.

If we set the conditions, the following equation will be drawn

$H(v, 0) = L(v) - L(u_0) = 0$

$H(v, 1) = L(u) + N(u) - f(r) = 0$

Now it can be written as a power series in p, as follows: $v = v_0 + pv_1 + p^2 v_2 + \ldots..$

If p =1, the best approximation is: $u = \lim_{p \rightarrow 1} v = v_0 + v_1 + v_2 + \ldots..$

III. COMPARISON OF SOLVING LAPLACE'S EQUATION BY SEPARATION OF VARIABLE METHOD AND HOMOTOPY PERTURBATION METHOD

The Laplace Equation is a second-order partial differential equation and it is denoted through the divergence symbol ∇. In free space, it is beneficial for deciding the electric-powered potentials. The Laplace equation is derived to make calculations in physics and arithmetic easier, as it is named after physicist Pierre-Simon Laplace.

The equation has the following form

$$\overrightarrow{\nabla^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots\ldots.. (3)$$

where u is the temperature defined as u(x, y) = X(x).Y (y) for separation of variable method.

Now, differentiating partially with respect to x and y we get,

$$\frac{\partial^2 u}{\partial x^2} = Y\frac{\partial^2 X}{\partial x^2} \qquad \frac{\partial^2 u}{\partial y^2} = X\frac{\partial^2 Y}{\partial y^2} \ldots\ldots\ldots\ldots (4)$$

So now using equation (3) in equation (4), we get

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = -\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = k \quad .... (5)$$

Case 1: If $k = \lambda^2 \geq 0$, we get the required solution as,
$U(x, y) = (A\,e^{\lambda x} + B\,e^{-\lambda x}) + (C\cos\lambda y + D\sin\lambda y)$

Case 2: If $k = -\lambda^2 \leq 0$, we get the required solution as,
$u(x, y) = (C\,e^{\lambda y} + D\,e^{-\lambda y}) + (A\cos\lambda x + B\sin\lambda x)$
where A, B, C, and D are arbitrary constants.

A. Solution of Laplace's Equation by Homotopy Perturbation Method

We start with Laplace's equation, with u as the temperature
$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with the following initial conditions
$u(0, y) = xf(y)$; $u(x, 0) = 0$ and $u_x(0, y) = f_1(y)$

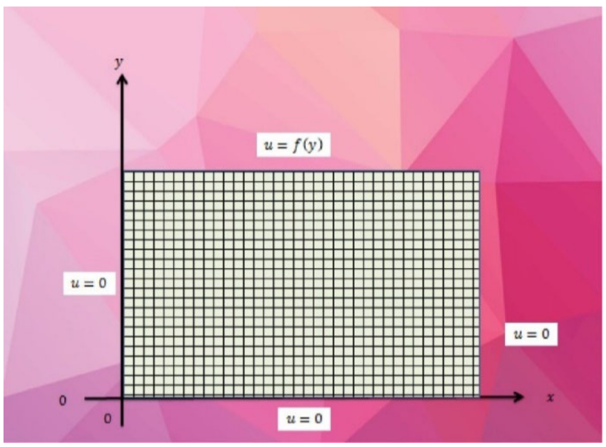

Fig. 2 Visualization of boundary condition through graphics

Applying the convex homotopy methods, we get

$$\frac{\partial^2}{\partial x^2}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots) + p\frac{\partial^2}{\partial y^2}(u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots) = 0$$

Now comparing the coefficients of equal power of p, we get

$$\frac{\partial^2 u_0}{\partial x^2} = 0$$

After integrating we get the solution as $u_0 = xh(y) + g(y)$
Now with the help of initial conditions we obtain

$$u_x = h(y) = f_1(y)$$

so the solution set becomes $u_0 = xf_1(y) + xf(y)$

Comparing the coefficients of p^1 and p^2 we get the expressions of Homotopy solution in the series format as follows

$$p^{(1)}: u_1(x, y) = -\frac{x^3}{6}[f_1{}^2(y) + f^2(y)] + u_0$$

Similarly,

$$p^{(2)}: u_2(x, y) = \frac{x^6}{720}[f_1{}^4(y) + f^4(y)] + \frac{x^2}{2}[f_1{}^2(y) + f^2(y)] + u_0$$

and in a similar manner we get, p^3, p^4, p^5 and so on, which leads us to write the general solution as follows

$$u = nu_0 + \sum_{i=1}^{n}\left\{(-1)^i \cdot x^{3i}\left(\frac{f_1{}^{2i}(y) + f^{2i}(y)}{(3i)!}\right)\right\}$$

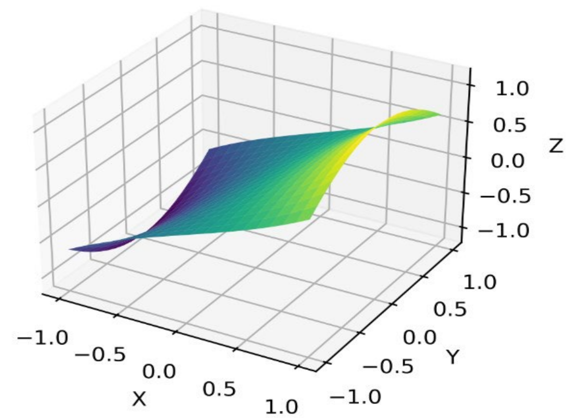

Fig. 3 3D Plot of Laplace's Equation using Homotopy Perturbation Method

The blessings of the Homotopy Perturbation Method have been proven in the above graphical representation. This is fairly a new technic and effortless to cope with for fixing linear and non-linear partial differential equations. It is an easy technique in contrast to some other iterative methods. For fixing this approach we get the nearest value of the actual solution than any other method. Error is much less than any other method.

TABLE I INITIAL CONDITION AND SOLUTION IN HPM METHOD

X	Y	Z	HPM
-1.0	-1.0	-1.0	-1.1750222
-0.8	-0.8	-0.8	-0.888019413
-0.6	-0.6	-0.6	-0.636618674
-0.4	-0.4	-0.4	-0.410742316
-0.2	-0.2	-0.2	-0.201334776
0.0	0.0	0.0	0.000000
0.2	0.2	0.2	0.20133477
0.4	0.4	0.4	0.410742316
0.6	0.6	0.6	0.636618674
0.8	0.8	0.8	0.888019413
1.0	1.0	1.0	1.1750222

IV. CONCLUSION

The modified Homotopy Perturbation Method (HPM) counselled right here is viewed to be one of the efficient techniques for acquiring the specific values for the Laplace equations. This technique can be prolonged for the Heat and Wave equation as well. As a result, this strategy is an extraordinarily tremendous mathematical instrument for solving any machine of partial differential equations. It is additionally a possible approach for fixing nonlinear partial differential equations. This HPM is additionally fitted for fixing Differential equations too. Through the method elaborated in [21], the ordinary differential equations can be solved with the usage of the HPM. The technique wishes much less work and charges very little (when in contrast with different numerical techniques like classical RK). In the bodily world, most occasions are taking place in a non-linear fashion. It is consequently very essential to learn about non-linear issues in the fields of physics, engineering, and different disciplines. But to get a precise answer to non-linear troubles is now not a convenient task. Often finding an approximate analytic answer is tougher than finding a numerical answer to non-linear problems. Many strategies are prescribed to resolve non-linear problems, such as the variational generation method, Homotopy perturbation method, etc. The HPM used to discover the answer into a framework of partial differential equations. The method is used directly, barring the use of linearization, transformation, discretization, or limiting assumptions. It is feasible to conclude that the HPM is extraordinarily effective and environment-friendly for discovering analytical options for an extensive variety of boundary price problems. It is a semi-analytic method. In physical issues, the strategy gives an extra sensible sequence of options that converge pretty quickly. It is well worth noting that the strategy can minimize the quantity of computing labor in contrast to normal strategies whilst keeping suitable numerical precision.

V. FUTURE SCOPE OF STUDY

This perturbation method (HPM) introduced in the learn about converges to the actual answer does not comprise any calculation of Adomian's polynomial, any linearization or discretization, any perturbation or transformation which may want to additionally enhance the accuracy of our model. This lookup can be prolonged to the dimensionless structure of Poisson's equation for the gravitational workability of a Newtonian self-gravitating, spherically symmetric, polytropic fluid. The venture with such fashions is that the ensuing gadget of the Lane Emden equation will be non-linear and in general analytical. This work can additionally be prolonged through a similar lookup to see what extra factors may additionally influence our technique closer to accuracy. We also have a plane to so the same in New Homotopy method.

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